

# Analysis of a Zero of a Beta Function Using All-Orders Summation of Diagrams

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Conventionally, one calculates a zero in a beta function by computing this function to a given loop order and solving for the zero. Here we discuss a different method which is applicable in theories where one can perform a partial diagrammatic summation to infinite-loop order. We show that this method, compared with the conventional method, yields much better agreement with exact results in the case of an asymptotically free gauge theory with  $\mathcal{N} = 1$  supersymmetry. Applications to other field theories are also discussed.

In a quantum field theory, the dependence of the interaction coupling on the Euclidean momentum scale,  $\mu$ , where it is measured is of fundamental importance. This dependence is described by the beta function,  $\beta$ , of the theory [1]. Of particular interest is a zero of  $\beta$ , since at this zero, the coupling is scale-invariant. We focus on a zero of  $\beta$  at nonvanishing coupling. The (physical) zero of  $\beta$  closest to the origin in coupling constant space is an infrared (IR) zero for an asymptotically free theory and an ultraviolet (UV) zero for an IR-free theory. In a conventional analysis of the beta function of a given theory, one considers the series expansion of  $\beta$  to a given order in powers of the coupling and investigates whether or not it has a (physical) zero away from the origin in coupling-constant space.

Here we discuss a different approach to the calculation of a zero of the beta function. Our approach is applicable to theories where one can perform a summation, to infinite-loop order, of the terms from a (sub)set of diagrams contributing to the beta function. Although our method can be applied to any theory where one can carry out this type of summation, it is conveniently illustrated in the context of an asymptotically free vectorial gauge theory. Thus, we consider such a gauge theory, with gauge group  $G$ , running coupling  $g = g(\mu)$ , and  $N_f$  massless Dirac fermions in a representation  $R$  of  $G$  [2]. Let  $\alpha(\mu) = g(\mu)^2/(4\pi)$ . If  $N_f$  is sufficiently large (still maintaining asymptotic freedom), then the two-loop beta function has an IR zero. As  $\mu$  decreases from large values in the UV,  $\alpha(\mu)$  increases toward the value of this zero, which thus plays an important role in determining the IR properties of the theory [3]. The beta function is  $\beta_g = dg/dt$ , where  $dt = d\ln \mu$ , or equivalently,  $\beta_\alpha = d\alpha/dt = [g/(2\pi)]\beta_g$ . This has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell a^\ell, \quad (1)$$

where  $b_\ell$  is the  $\ell$ -loop coefficient and  $a = g^2/(16\pi^2)$ . The coefficients  $b_1$  and  $b_2$  are independent of the regularization scheme, while the  $b_\ell$  for  $\ell \geq 3$  are scheme-dependent [4]. The property of asymptotic freedom means  $b_1 > 0$ . The  $n$ -loop ( $n\ell$ ) beta function, denoted  $\beta_{\alpha,n\ell}$ , is given by Eq. (1) with  $\ell = n$  as the upper limit on the sum. We

define the reduced ( $r$ ) function

$$\beta_{\alpha,r} \equiv -\frac{\beta_\alpha}{[\alpha^2 b_1/(2\pi)]} = 1 + \frac{1}{b_1} \sum_{\ell=2}^{\infty} b_\ell a^{\ell-1} \quad (2)$$

and  $\beta_{\alpha,r,n\ell}$ , given by Eq. (2) with the upper limit  $\ell = n$  on the sum. The conventional approach to the study of a possible IR zero in  $\beta_{\alpha,n\ell}$  consists of an analysis of the  $n-1$  zeros of the  $n$ -loop reduced beta function  $\beta_{\alpha,r,n\ell}$ . In particular, at the two-loop level, the IR zero is given by  $\alpha_{IR,2\ell} = -4\pi b_1/b_2$ , which is physical if  $b_2 < 0$ . Since we are interested in an IR zero here, we thus assume that the matter content of the theory is such that  $b_2 < 0$ .

However, if one can carry out a summation of an infinite (sub)set of diagrams contributing to the beta function, this yields a result that, although formally equivalent to (1) when expanded in a series, involves a closed-form (*cf*) functional factor,  $f_{cf}$ . Normalizing this factor so that  $f_{cf} = 1$  at  $a = 0$ , we can reexpress  $\beta_\alpha$  in this case as

$$\beta_\alpha = -\frac{\alpha^2 b_1}{2\pi} f_{cf} f_s, \quad (3)$$

i.e.,  $\beta_{\alpha,r} = f_{cf} f_s$ , where  $f_s$  is given by a series ( $s$ ) expansion

$$f_s = 1 + \sum_{j=1}^{\infty} f_{s,j} a^j. \quad (4)$$

In a theory where this summation can be carried out and  $\beta_\alpha$  can thus be expressed in the form (3), our proposed method is to analyze the zeros in  $f_{cf}$  and  $f_s$  rather than the zeros of  $\beta_{\alpha,r,n\ell}$  as a means of gaining information about the physical zero in the beta function. This method takes advantage of the information from the all-orders summation, while the conventional method of analyzing zeros in  $\beta_{\alpha,r,n\ell}$  to a given finite loop order  $n$  does not. In particular, if  $f_{cf}$  has no physical zero, then the proposed method is to examine the zeros in  $f_s$ .

Let us consider an illustrative function for  $f_{cf}$ , namely

$$f_{cf} = \frac{1}{1 + d_1 a}, \quad (5)$$

(where  $d_1$  is a constant), so that  $\beta_{\alpha,r} = f_s/(1 + d_1 a)$ . Then

$$f_s = 1 + \frac{1}{b_1} \sum_{n=2}^{\infty} (b_n + d_1 b_{n-1}) a^{n-1}. \quad (6)$$

Solving the equation  $f_s = 0$  to the lowest nontrivial order,  $n = 2$ , we obtain for the IR zero

$$\alpha_{IR,cfs} = -\frac{4\pi}{(b_2/b_1) + d_1}, \quad (7)$$

which is physical if  $(b_2/b_1) + d_1 < 0$  (the subscript  $cfs$  connotes the direct dependence on  $f_s$  and the implicit dependence on  $f_{cf}$ .) Since we assume  $b_2 < 0$  (and  $b_1 > 0$ ),  $\alpha_{IR,cfs}$  is physical if

$$d_1 < \frac{|b_2|}{b_1}. \quad (8)$$

Because the existence of the IR zero in  $\beta_{\alpha,IR,2\ell}$  is scheme-independent, one may require that the scheme used for the summation to infinite-loop order that yields (3) should not remove this IR zero, and hence that  $d_1$  satisfies the inequality (8). As compared with  $\alpha_{IR,2\ell}$ ,

$$\alpha_{IR,cfs} = \frac{\alpha_{IR,2\ell}}{1 - d_1 \frac{a_{IR,2\ell}}{4\pi}}. \quad (9)$$

Hence, if  $d_1$  is negative (positive), then  $\alpha_{IR,cfs}$  is smaller (larger) than  $\alpha_{IR,2\ell}$ . As is evident from the dependence on  $d_1$ ,  $\alpha_{IR,cfs}$  incorporates information from the summation to infinite-loop order, in contrast to  $\alpha_{IR,n\ell}$  for any finite  $n$ . A necessary condition for the result (9) to be physically meaningful is that if  $f_{cf}$  has any divergent physical singularities, they must occur farther from the origin than  $\alpha_{IR,cfs}$ . In our example,  $f_{cf}$  has a simple pole at  $\alpha_{cf,pole} = -4\pi/d_1$ , which occurs at physical coupling if  $d_1 < 0$ . The required inequality is  $\alpha_{cf,pole} > \alpha_{IR,cfs}$ , i.e.,

$$-\frac{1}{d_1} > -\frac{1}{(b_2/b_1) + d_1}. \quad (10)$$

Since  $b_2 < 0$  for the existence of  $\alpha_{IR,2\ell}$  and we only need to address the case where  $d_1$  is negative, (10) is equivalent to the inequality

$$\frac{1}{|d_1|} > \frac{1}{(|b_2|/b_1) + |d_1|}, \quad (11)$$

which is clearly true. This proves that (for the case we consider here, where the theory has an IR zero in  $\beta_{\alpha,2\ell}$ ) even if  $f_{cf}$  contains a physical pole, one has  $\alpha_{cf,pole} > \alpha_{IR,cfs}$ , so this pole is irrelevant for the evolution from weak coupling in the UV to the IR zero at  $\alpha_{IR,cfs}$ .

To demonstrate the usefulness of the proposed method, we apply it to an asymptotically free vectorial  $\mathcal{N} = 1$  supersymmetric gauge theory (denoted SGT) with gauge group  $SU(N_c)$  and  $N_f$  pairs of massless chiral superfields  $\Phi_j, \tilde{\Phi}_j$ ,  $1 \leq j \leq N_f$  transforming according to the representations  $R$  and  $\bar{R}$  of  $SU(N_c)$ , respectively. In this theory, the scheme-independent coefficients in  $\beta_\alpha$  are  $b_1 = 3C_A - 2T_f N_f$  and  $b_2 = 6C_A^2 - 4(C_A + 2C_f)T_f N_f$  [5, 6]. Calculations to three-loop [7] order have been done, e.g., in the  $\overline{DR}$  scheme [8]. This theory has the

appeal that a number of exact results are known for it [9]–[11], enabling one to compare approximate calculations with these exact results [12]–[14]. We will show that the method discussed here, compared with the conventional method, yields much better agreement with exact results and solves a puzzle found in [13].

In this SGT the constraint of asymptotic freedom requires that  $N_f < N_{f,b1z}$ , where  $N_{f,b1z} = 3C_A/(2T_f)$  [15]. It is known [9]–[11] that if  $N_f$  is in the interval  $I_{NACP}$  defined by  $N_{f,cr} < N_f < N_{f,b1z}$ , where

$$N_{f,cr} = \frac{3C_A}{4T_f} = \frac{N_{f,b1z}}{2}, \quad (12)$$

then this theory flows from weak coupling in the UV to a conformal non-Abelian Coulomb phase (NACP) in the IR without any spontaneous chiral symmetry breaking ( $S\chi SB$ ). A quantity of interest is the anomalous dimension of the fermion bilinear,  $\gamma_m$  [16]. This has the series expansion

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell a^\ell, \quad (13)$$

where  $c_1 = 4C_f$  is scheme-independent, and the  $c_\ell$  with  $\ell \geq 2$  are scheme-dependent. Using instanton methods, Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) obtained an exact closed-form solution for  $\beta_\alpha$  [9, 10] (in a particular scheme [17]):

$$\beta_{\alpha,NSVZ} = -\frac{\alpha^2}{2\pi} \left[ \frac{b_1 - 2T_f N_f \gamma_m}{1 - 2C_A a} \right]. \quad (14)$$

As  $N_f$  decreases from  $N_{f,b1z}$  and approaches  $N_{f,cr}$  from above,  $\gamma_m$  increases from 0 to the maximum value allowed by unitarity in this conformal phase, namely  $\gamma_m = 1$  [18, 19]. Applying the condition that  $\gamma_m$  should saturate this unitarity upper bound at an IR zero of  $\beta$  as  $N_f$  approaches  $N_{f,cr}$  from above is one way to derive the value of  $N_{f,cr}$ .

Using known results for the  $b_\ell$  and  $c_\ell$  for  $1 \leq \ell \leq 3$  [5]–[7], Ref. [13] calculated  $\alpha_{IR,n\ell}$  at the  $n = 2$  and  $n = 3$  loop level and evaluated the  $n$ -loop anomalous dimension of the fermion bilinear at  $\alpha = \alpha_{IR,n\ell}$ , denoted  $\gamma_{IR,n\ell}$ . Although higher-order calculations of  $\alpha_{IR,n\ell}$  are scheme-dependent, they are quite valuable in obtaining more accurate information about  $\gamma_m$  at an exact IR fixed point, which is a universal quantity. Ref. [13] found that at the two-loop level,  $\gamma_{IR,2\ell}$  increases monotonically as  $N_f$  decreases from  $N_{f,b1z}$ , but exceeds unity at a value of  $N_f \in I_{NACP}$  larger than  $N_{f,cr}$ . As was noted in [13], this is understandable in view of the fact that the two-loop perturbative calculations of  $\beta$  and  $\gamma_m$  are not expected to agree precisely with exact results; the differences provide a quantitative measure of the accuracy of these two-loop calculations. Normally, one would expect that calculating a quantity to higher-loop order should give a more accurate result if perturbative computations are reliable. However, the study of the IR behavior at

three-loop level in [13] yielded a different and very puzzling result:  $\gamma_{IR,3\ell}$  does not approach more closely to 1 as  $N_f$  decreases through  $I_{NACP}$  toward  $N_{f,cr}$ ; on the contrary,  $\gamma_{IR,3\ell}$  reaches a maximum at a small positive value and then decreases, passing through zero to negative values.

We next show how our proposed method removes this puzzling behavior and produces excellent agreement with exact results for this SGT. First, observe that  $\beta_{\alpha,NSVZ}$  has the form of Eq. (3) with

$$f_{cf,NSVZ} = \frac{1}{1 - 2C_A a}, \quad (15)$$

and

$$f_{s,NSVZ} = 1 - \frac{2T_f N_f \gamma_m}{b_1}, \quad (16)$$

We focus on the range of  $N_f$  where  $\beta_{\alpha,NSVZ}$  has an IR zero. Applying our method, we observe that  $f_{cf,NSVZ}$  has no zero, so we solve the equation  $f_{s,NSVZ} = 0$ . Retaining the maximal scheme-independent information in  $\gamma_m$ , namely the  $\ell = 1$  term in (13), this equation reads  $b_1 - 8N_f T_f C_f a = 0$ , whence

$$\alpha_{IR,cfs} = \frac{\pi(3C_A - 2T_f N_f)}{2T_f C_f N_f} = \frac{\pi}{C_f} \left[ \frac{2N_{f,cr}}{N_f} - 1 \right]. \quad (17)$$

As  $N_f$  decreases from  $N_{f,b1z}$  to  $N_{f,cr}$ ,  $\alpha_{IR,cfs}$  increases monotonically from 0 to  $\alpha_{cfs} = \pi/C_f$ . Since  $d_1 < 0$  (and with  $b_2 < 0$  so that  $\alpha_{IR,2\ell}$  is physical), it follows, as a special case of (9), that,  $\alpha_{IR,cfs} < \alpha_{IR,2\ell}$  in this SGT. Our general discussion above shows that the pole in  $f_{cf,NSVZ}$  at  $\alpha_{cf,pole,NSVZ} = 2\pi/C_A$  lies farther from the origin than  $\alpha_{IR,cfs}$  [20]. As  $N_f$  approaches  $N_{f,b1z}$  from below, both  $\alpha_{IR,cfs}$  and  $\alpha_{IR,2\ell}$  approach zero, and the ratio  $\alpha_{IR,cfs}/\alpha_{IR,2\ell} \nearrow 1$ .

Thus,  $\alpha_{IR,cfs}$  retains perturbative reliability much better than the conventional two-loop result

$$\alpha_{IR,2\ell} = \frac{2\pi(3C_A - 2T_f N_f)}{2(C_A + 2C_f)T_f N_f - 3C_A^2}, \quad (18)$$

which diverges as  $N_f \searrow N_{f,b2z} = 3C_A^2/[2(C_A + 2C_f)T_f]$ , where  $b_2 \rightarrow 0$ . For  $R$  equal to the fundamental representation (with Young tableau  $\square$ ),  $N_{f,b2z} \in I_{NACP}$ .

Next, we study  $\gamma_m$ , calculated to its maximal scheme-independent order,  $\gamma_m = 4C_f a$ , and evaluated at  $\alpha_{IR,cfs}$ . We obtain

$$\gamma_{IR,cfs} = \frac{3C_A}{2T_f N_f} - 1 = \frac{N_{f,b1z}}{N_f} - 1 = \frac{2N_{f,cr}}{N_f} - 1. \quad (19)$$

From (19), it follows that  $\gamma_m \searrow 0$  as  $N_f \nearrow N_{f,b1z}$ . Furthermore, from (19) we obtain three important results for  $\gamma_m$ , which exhibit complete agreement with exact results and avoid the problems with  $\gamma_{IR,2\ell}$  and  $\gamma_{IR,3\ell}$  found in [13]. First, Eq. (19) shows that as  $N_f$  decreases from  $N_{f,b1z}$  to  $N_{f,cr}$  in the interval  $I_{NACP}$ ,  $\gamma_{IR,cfs}$  increases monotonically. In contrast,  $\gamma_{IR,3\ell}$  was found not

to be a monotonically increasing function with decreasing  $N_f \in I_{NACP}$  [13].

Second,  $\gamma_{IR,cfs} \leq 1$  for all  $N_f \in I_{NACP}$ , in agreement with the constraint from unitarity. In contrast,  $\gamma_{IR,2\ell} =$  was found to violate this unitarity constraint for a range of  $N_f$  toward the lower end of the interval  $I_{NACP}$  [13]; indeed, it was found that  $\gamma_{IR,2\ell}$  diverges, just as  $\alpha_{IR,2\ell}$  does, as  $N_f \searrow N_{f,b2z}$  [13]. As was noted in [13], this behavior indicated that the perturbative calculations of  $\alpha_{IR,2\ell}$  and the resultant evaluation of  $\gamma_m$  at this value of  $\alpha_{IR,2\ell}$  are not reliable in this region. With our different procedure, we avoid this pathological behavior.

Third, we find that

$$\gamma_{IR,cfs} \nearrow 1 \text{ as } N_f \searrow N_{f,cr}. \quad (20)$$

Thus, this  $\gamma_{IR,cfs}$  has precisely the three properties of the exact all-orders calculation of  $\gamma_{IR}$ . Therefore, we have shown that our method of calculating the IR zero of the beta function as a zero of  $f_s$  rather than as the zero of  $\beta_{\alpha,r,n\ell}$ , together with the evaluation of  $\gamma_m$  calculated to its maximal scheme-independent order to obtain  $\gamma_{IR,cfs}$ , yields excellent agreement with exact results and avoids the problems with  $\gamma_{IR,2\ell}$  and  $\gamma_{IR,3\ell}$  found in [13].

For the case  $R = \square$ , Eqs. (17) and (19) are

$$\alpha_{IR,cfs,\square} = \frac{2\pi N_c(3N_c - N_f)}{(N_c^2 - 1)N_f} \quad (21)$$

and

$$\gamma_{IR,cfs,\square} = \frac{3N_c}{N_f} - 1. \quad (22)$$

In this  $R = \square$  case, it is also of interest to consider the 't Hooft-Veneziano limit  $N_c \rightarrow \infty$  and  $N_f \rightarrow \infty$  with  $r = N_f/N_c$  fixed and finite, and  $\alpha(\mu)N_c$  a finite function of  $\mu$ . In this limit,  $I_{NACP}$  is  $3/2 \leq r \leq 3$ . In terms of  $\xi \equiv \alpha N_c$  in this limit, we have

$$\xi_{IR,cfs} = \frac{2\pi(3-r)}{r}, \quad \gamma_{IR,cfs} = \frac{3-r}{r}. \quad (23)$$

In contrast,

$$\xi_{IR,2\ell} = \frac{2\pi(3-r)}{2r-3}, \quad (24)$$

which diverges as  $r \searrow 3/2$  at the lower boundary of  $I_{NACP}$ , and

$$\gamma_{IR,2\ell} = \frac{r(r-1)(3-r)}{2(3-2r)^2}, \quad (25)$$

which violates its unitarity upper bound as  $r$  decreases through  $r = 2$  in the interior of  $I_{NACP}$  [13, 14] and diverges as  $r \searrow 3/2$ .

Since the  $b_\ell$  with  $\ell \geq 3$  are scheme-dependent, so is the closed-form function  $f_{cf}$  resulting from the infinite-loop summation. One might think that the analysis of an IR zero in  $\beta_\alpha$  could be simplified by applying a scheme

transformation (ST) to eliminate the  $b_\ell$  with  $\ell \geq 3$ . An explicit ST that reduces  $\beta_\alpha$  to the two-loop expression in the local neighborhood of the origin,  $\alpha = 0$ , was constructed and studied in [21]-[23] (see also [24]). However, it was shown that STs that are acceptable for small  $\alpha$  often lead to unphysical results when applied at a generic zero of a beta function away from the origin [21]-[23]. We find this to be true for STs that attempt to remove  $b_\ell$  with  $\ell \geq 3$  in the present SGT. Hence, one must deal with the full series (1), where the method discussed here greatly improves the analysis.

Another demonstration of the usefulness of the method discussed here is obtained from analysis of a test function for  $\beta_\alpha$ . A basic question for a study of the zero(s) of a beta function and, more generally, in mathematics is: how well do the zero(s) of a truncation of a power series for a function  $f(z)$  reproduce the zero(s) of the exact function. In Ref. [22] this question was investigated for a general asymptotically free gauge theory, using for  $\beta_{\alpha,r}$  the test function

$$\beta_{\alpha,r} = h(\alpha) = \frac{\sin(\pi\sqrt{\tilde{\alpha}})}{(\pi\sqrt{\tilde{\alpha}})}, \quad (26)$$

where  $\tilde{\alpha} \equiv \alpha/\alpha_{IR}$ . We analyzed the zeros of truncations of the Taylor series expansion of  $h(\alpha)$  to high order and determined how rapidly these approach  $\alpha_{IR}$ , i.e., how rapidly the zeros in the ratios  $\tilde{\alpha}$  approach 1. Here, to apply our method, we start from the above series for  $h(\alpha)$  and perform an infinite-order summation of part of it, using the identity  $(\sin x)/x = [(2/x)\sin(x/2)]\cos(x/2)$  with  $x = \pi\sqrt{\tilde{\alpha}}$ , taking  $(2/x)\sin(x/2) = f_{cf}$  to be the closed-form function and taking  $\cos(x/2) = f_s$ , expanded as a series. That is, we write

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \left[ \frac{\sin(x/2)}{x/2} \right] \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(2n)!} \quad (27)$$

This  $f_{cf}$  has no zero in the interval  $0 \leq x \leq \pi$ , i.e.,  $0 \leq \alpha \leq \alpha_{IR}$ , so the IR zero in  $\beta_{\alpha,r}$  comes from  $f_s$ . We calculate this from the truncation of the series for  $f_s$  and denote it as  $\alpha_{cfs,n}$ . In Table I we list the values of  $\delta_n \equiv \tilde{\alpha}_{IR,n} - 1$  obtained by (i) the conventional method of analyzing  $\beta_{\alpha,r,n}$  as a series, as in [22], denoted  $(\delta_n)_h$ , and by (ii) the method discussed here, with  $(\delta_n)_{cfs} = \tilde{\alpha}_{cfs,n} - 1$ . As is evident, our present method yields faster convergence toward the exact result.

For a non-supersymmetric gauge theory (in  $d = 4$  dimensions) with  $N_f$  fermions in a general representation  $R$  of the gauge group  $G$ , a rigorous closed-form expression for  $\beta_\alpha$  has not, so far, been obtained, although a conjecture for an all-orders beta function has been presented in [25]. Higher-loop analyses of an IR zero in  $\beta_\alpha$  have been performed using the conventional method of studying the zeros of the polynomial  $\beta_{\alpha,r,n\ell}$  up to the four-loop level [26, 27]. These obtained the expected result that calculating  $\alpha_{IR,n\ell}$  and  $\gamma_{IR,n\ell}$  to higher-loop order led to more stable results. For example, for  $G = \text{SU}(3)$  and  $N_f = 12$

TABLE I: Values of  $(\delta_n)_h$  and  $(\delta_n)_{cfs}$  as a function of respective series truncation order  $n$ .

$n$	$(\delta_n)_h$	$(\delta_n)_{cfs}$
2	-0.392	-0.189
3	complex	$2.78 \times 10^{-2}$
4	$-3.97 \times 10^{-2}$	$-1.13 \times 10^{-3}$
5	$4.52 \times 10^{-3}$	$3.15 \times 10^{-5}$
6	$-2.83 \times 10^{-4}$	$-0.592 \times 10^{-6}$
7	$1.35 \times 10^{-5}$	$0.805 \times 10^{-8}$
8	$-0.493 \times 10^{-6}$	$-0.829 \times 10^{-10}$

fermions in  $R = \square$ ,  $\gamma_{IR,2\ell} = 0.77$ ,  $\gamma_{IR,3\ell} = 0.31$  and  $\gamma_{IR,4\ell} = 0.25$  [27]. The four-loop result agrees well with the lattice measurements  $\gamma = 0.27(3)$  and  $\gamma_{IR} \simeq 0.25$  [28] and  $\gamma = 0.235(46)$  [29][30].

There are also IR-free theories where one can obtain an exact closed-form result for the beta function with a UV zero at nonzero coupling. A first example is the  $N \rightarrow \infty$  limit of the nonlinear  $O(N)$   $\sigma$  model in  $d = 2 + \epsilon$  dimensions[31]. Here, for small  $\epsilon$ , one finds the closed-form result [31]

$$\beta_x = \frac{dx}{dt} = \epsilon x \left( 1 - \frac{x}{x_c} \right), \quad (28)$$

where  $\lambda$  is the effective coupling,  $\lambda_c = 2\pi\epsilon/N$ ;  $x = \lim_{N \rightarrow \infty} \lambda N$ , and  $x_c = 2\pi\epsilon$  is the UV zero of  $\beta_x$ . In the notation of Eq. (3), here  $f_{cf} = 1 - (x/x_c)$  and  $f_s = 1$ . A second example is supersymmetric quantum electrodynamics (SQED) with  $N_f$  pairs of chiral superfields with charges that can be taken to be 1 and  $-1$ . As is well known,  $\beta_\alpha$  for this theory can be obtained from (14) by setting  $C_A = 0$ ,  $T_f = 1$ , and  $C_f = 1$ , and is [32]  $\beta_{\alpha,SQED} = [\alpha^2 N_f / (2\pi)] [1 + (\alpha/\pi)]$ . Up to the reversal of sign, this has the form of Eq. (3) with  $f_{cf} = 1 + (\alpha/\pi)$  and  $f_s = 1$ . It has no UV zero. Third, one may consider the SGT with  $N_f > N_{f,b1z}$ . Using (14) again for this theory, one observes that the numerator of (14) is  $-|b_1| - 2N_f T_f \gamma_m$ . Evaluating  $\gamma_m$  at the maximal scheme-independent level,  $\gamma_m = 4C_f a$ , it follows that this numerator has no UV zero. Some related work for a nonsupersymmetric  $U(1)$  gauge theory (in  $d = 4$ ) is in [33, 34]; in this theory, both a conventional analysis of  $\beta_{\alpha,r}$  up to the five-loop level and use of some partial closed-form results also led to evidence against a UV zero [34].

In summary, we have discussed a method for calculating a zero in the beta function of a quantum field theory that is applicable in cases where one can perform an infinite-loop summation of a subset of diagrams. We have shown that this method, compared with the conventional fixed-loop-order method, yields much better agreement with exact results in the case of an asymptotically free gauge theory with  $\mathcal{N} = 1$  supersymmetry. We have also commented on the application of this method to some

IR-free theories.

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